Solution to Problem 14) Wallis's product formula may be rearranged, as follows:

$$\frac{\pi}{2} = \lim_{m\to\infty} \frac{(2m)!!}{(2m-1)!!} \times \frac{(2m)!!}{(2m+1)!!} = \lim_{m\to\infty} \frac{[(2m)!!]^4}{(2m)! \times (2m+1)!} = \lim_{m\to\infty} \frac{2^{4m}(m!)^4}{(2m+1) \times [(2m)!]^2}.$$

In the limit when $m \to \infty$, substitution of Stirling's asymptotic formula for $m!$ and $(2m)!$ in the above equation yields

$$\frac{\pi}{2} = \lim_{m\to\infty} \frac{2^{4m}[c\sqrt{m}\,(m/e)^m]^4}{(2m+1) \times [c\sqrt{2m}\,(2m/e)^{2m}]^2} = c^2 \lim_{m\to\infty} \frac{2^{4m}m^2(m/e)^{4m}}{(2m+1) \times 2m(2m/e)^{4m}}$$

$$= c^2 \lim_{m\to\infty} \frac{m}{2(2m+1)} = \frac{c^2}{4} \qquad \rightarrow \qquad c = \sqrt{2\pi}.$$

Digression: Strictly speaking, one needs to demonstrate that $n!/[\sqrt{n}(n/e)^n]$ approaches a limit when $n \to \infty$, before one can assign a constant c to this limit. Given that Stirling's approximation has already established an upper bound, e, and a lower bound, $e^{7/8}$, for the ratio $n!/[\sqrt{n}(n/e)^n]$, it suffices to verify that the sequence is either monotonically increasing or monotonically decreasing as $n \to \infty$. Recalling that the logarithmic function is monotonic, we examine the sequence $\alpha_n = \ln\{n!/[\sqrt{n}(n/e)^n]\}$ for its monotonicity.

$$\alpha_{n+1} - \alpha_n = \ln[(n+1)!] - \ln\sqrt{n+1} - (n+1)\ln[(n+1)/e] - [\ln(n!) - \ln\sqrt{n} - n\ln(n/e)]$$

$$= \ln(n+1) - \tfrac{1}{2}\ln(1+n^{-1}) - n\ln(1+n^{-1}) - \ln(n+1) + 1$$

$$= 1 - (n+\tfrac{1}{2})\ln(1+n^{-1})$$

$$= 1 - \tfrac{1}{2}(2n+1)\left[\ln\left(1+\frac{1}{2n+1}\right) - \ln\left(1-\frac{1}{2n+1}\right)\right]$$

$$= 1 - \tfrac{1}{2}(2n+1)\left[\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(2n+1)^k} + \sum_{k=1}^{\infty} \frac{1}{k(2n+1)^k}\right]$$

$$= 1 - \tfrac{1}{2}(2n+1)\sum_{k=0}^{\infty} \frac{2}{(2k+1)(2n+1)^{2k+1}}$$

$$= 1 - \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2n+1)^{2k}} = -\sum_{k=1}^{\infty} \frac{1}{(2k+1)(2n+1)^{2k}}$$

Clearly, $\alpha_{n+1} - \alpha_n < 0$, which indicates that the sequence is monotonically decreasing.
